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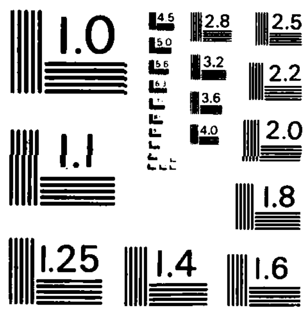

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Final Report

ANALYTICAL MODEL OF AURORA CURRENTS

By

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I. INTRODUCTION

This report deals with the current flow of Birkland currents injected into the auroral zone by solar events and penetrating through the ionosphere, strato- and troposphere to the earth surface. In the calculation of the aurora currents, the following simplifying assumptions have been made. The earth's magnetic field is constant and vertical with respect to the earth surface. The problem is two dimensional. The potential function and the electric field are functions of the vertical coordinate z and one horizontal coordinate x of a Cartesian coordinate system x, y, z. The Hall current flows only in the y direction. Steady state conditions are assumed, that is, the electric functions are independent of time. The conductivity is scalar and exponential in the tropo- and stratosphere. In the ionosphere, the parallel conductivity continues to increase exponentially, whereas the Pedersen or transverse conductivity decreases. These conductivity profiles are expressed by suitable analytic functions.

The earth is assumed to be a plain equipotential surface. The current is injected into the ionosphere vertically in a strip of 75 km width and ejected in a parallel strip of the same width. The solution of the differential equation is given in harmonics or eigenfunctions.

This solution may be considered as a first example to show a way how these type of problems can be solved analytically. The restrictions may be relaxed by accommodating different conductivity profiles, extension to time dependent problems, and maybe to different coordinate systems.

II. SOLUTION OF DIFFERENTIAL EQUATION.

The following symbols and definitions are used in this calculation:

x, y, z = Cartesian coordinate system with the z axis pointing upwards.

$\bar{i}, \bar{j}, \bar{k}$ = Unit vectors in direction of the coordinates

$\lambda_1 = \lambda_0 \exp(2kz)$ = scalar and parallel conductivity

$\lambda_0 = 2 \times 10^{-14}\, 1/\Omega m$ = conductivity at the ground

$2k = \frac{\ln 10}{10\ \mathrm{km}}$ = scale factor of conductivity

$\lambda_2 = \lambda_1/(1+a \exp(2kz))^2$ = Pedersen or transverse conductivity.

$\phi = \phi(x,z)$ = Potential function

$\bar{E} = -\nabla\phi$ = electric field

$\bar{I} = \bar{\bar{L}}\,\bar{E}$ ($\bar{\bar{L}}$ is the conductivity tensor in the ionosphere and a scalar in the strato- and troposphere)

$\bar{H}$ = magnetic field vector

μ = permeability.

A-1

Maxwell's equations

$$\nabla \times \bar{H} = \bar{I} \qquad (1)$$

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \qquad (2)$$

Since the current flow is assumed to be in steady state from (2)

$$\frac{\partial H}{\partial t} = 0; \quad \nabla \times \bar{E} = 0; \ \bar{E} = -\nabla\phi \qquad (3)$$

and taking the divergence of (1)

$$\nabla \cdot \nabla \times \bar{H} = 0; \ \nabla \cdot \bar{I} = -\nabla\cdot(\bar{i}\lambda_2 \frac{\partial\phi}{\partial x} + \bar{j}\lambda_3(\bar{i}\frac{\partial\phi}{\partial x} \times \bar{B})/B^2$$

$$+ \bar{k}\lambda_1 \frac{\partial\phi}{\partial z}) = 0 \qquad (4)$$

The divergence of the Hall current in (4) is zero because it requires differentiation to y and the Hall current of our problem is not a function of y. Executing the divergence operation in (4) we obtain the differential equation of our problem

$$\frac{\partial}{\partial x}\lambda_2 \frac{\partial\phi}{\partial x} + \frac{\partial}{\partial z}\lambda_1 \frac{\partial\phi}{\partial z} = 0 \qquad (5)$$

Since λ_2 is a function of z only, we may put it before the differential operator to x and divide (5) by λ_2

$$\frac{\partial^2\phi}{\partial x^2} + \frac{1}{\lambda_2} \frac{\partial}{\partial z}\lambda_1 \frac{\partial\phi}{\partial z} = 0 \qquad (6)$$

This form of the differential equation (6) suggests immediately a solution in the form of eigenfunctions or harmonics. We assume that ϕ is given as a product of a function $f = f(z)$ and $g = g(x)$

$$\phi = g(x)\ f(z) \tag{7}$$

By writing the factor not affected by the differentiation before the differential operator and dividing by ϕ splits (6) into two terms, the first depending only on x and the second on z.

$$\frac{1}{g}\frac{d^2g}{dx^2} + \frac{1}{\lambda_2 f}\frac{d}{dz}\lambda_1\frac{df}{dz} = 0 \tag{8}$$

We set now the first term equal to the arbitrary constant $-s^2$.

$$\frac{1}{g}\frac{d^2g}{dx^2} = -s^2 \tag{9}$$

with the solution

$$g = A\ \exp(\pm isx)\ ;\quad i = \sqrt{-1} \tag{10}$$

A is here a complex integration constant and s a scaling factor of x.

Replacing the first term in (8) by $-s^2$ we obtain for f the differential equation

$$\frac{d}{dz}\lambda_1\ \frac{df}{dz} - s^2\lambda_2'\ f = 0 \tag{11}$$

With

$$\frac{d\lambda_1}{dz} = 2k\lambda_1, \quad \lambda_2 = \lambda_1/(1 + a \exp 2kz)^2$$

expanding the first term in (11) results in

$$\lambda_1 \frac{d^2f}{dz^2} + 2k\lambda_1 \frac{df}{dz} - s^2 \frac{\lambda_1}{(1 + a \exp 2kz)^2} f = 0$$

The function λ_1 drops out and we may simplify the equation further by the substitution

$$u = \exp(-2kz); \quad \frac{du}{dz} = -2ku \tag{12}$$

We obtain then the simple differential equation

$$\frac{d^2f}{du^2} - \left(\frac{s}{2k}\right)^2 \frac{f}{(u+a)^2} = 0 \tag{13}$$

We set now

$$f = (u+a)^m \tag{14}$$

$$\frac{d^2f}{du^2} = m(m-1) \ (u+a)^{m-2} = m(m-1) \frac{f}{(u+a)^2} \tag{15}$$

and substitute expression (15) in (13). This gives us a quadratic equation for m

$$m^2 - m - \left(\frac{s}{2k}\right)^2 = 0 \tag{16}$$

with the two solutions for m

$$m_1 = \frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{s}{2k}\right)^2} = \frac{1}{2}\left(\sqrt{1 + (s/k)^2} + 1\right) \tag{17}$$

$$m_2 = \frac{1}{2} - \sqrt{\frac{1}{4} + \left(\frac{s}{2k}\right)^2} = -\frac{1}{2}\left(\sqrt{1 + (s/k)^2} - 1\right) \tag{18}$$

and the two solutions for f

$$f_1 = (\exp(-2kz) + a)^{m_1} \tag{19}$$

$$f_2 = (\exp(-2kz) + a)^{m_2} \tag{20}$$

With the solution for g given in (10) we obtain from (10), (19), and (20) the two potential functions

$$\phi_1 = A \exp(\pm isx)\ f_1 \tag{21}$$

$$\phi_2 = B \exp(\pm isx)\ f_2 \tag{22}$$

A and B are complex arbitrary integration constants that can be used to adjust the solution to given boundary conditions. This will be discussed in more detail in Section IV.

III. DISCUSSION OF THE SOLUTION

The functions ϕ_1 and ϕ_2 represent the in and out pole functions of the differential equation, meaning that the inpole function ϕ_1 is caused by a source at the ground ($z = 0$), where ϕ_1 has its maximum value that decreases with increasing distance from the source. The outpole function ϕ_2 is produced by a source at some altitude H

above ground, for instance in the ionosphere or above, has again its maximum value at the source ($z = H$) that decreases with increasing distance from the source, i.e. as we approach the ground ($z = 0$). This distinction between in and out pole potential functions is reflected in modern terminology as up (source below) or down (source above) mapping.

To exclude the ionosphere and restrict the solution to the tropo- and stratosphere where $\lambda_1 = \lambda_2$ we set $a = 0$. The solution for g is not affected and remains the same as given in (10). However, f_1 and f_2 reduce to the simpler form

$$f_1 = e^{-m_1 2kz} \tag{23}$$

$$f_2 = e^{-m_2 2kz} \tag{24}$$

resulting in

$$\phi_1 = A \exp(\pm isx) \exp(-(\sqrt{1 + (s/k)^2} + 1)kz) \tag{25}$$

$$\phi_2 = A \exp(\pm isx) \exp((\sqrt{1 + (s/k)^2} - 1)kz) \tag{26}$$

In this reduced form it is easy to recognize that ϕ_1 is the inpole function with its maximum value at the ground $z = 0$ and diminishing with increasing z. ϕ_2 is the outpole function, that has its minimum value at the ground and increases with increasing z. Furthermore, we see that the inpole potential decreases much faster than

the outpole potential increases on looking from the source that the attenuation by mapping down is much smaller than the attenuation by mapping up, a fact which is often mentioned in the literature. The analytical solution gives the additional information that the weak attenuation by down mapping depends also on the ratio of s/k. If this ratio is much larger than 1 meaning for wide spread sources the attenuation by up and down mapping will be of the same order.

We may reduce (25) and (26) further by multiplying the bracket in the second exp-function with k and then let $k \to 0$. This would reduce the conductivity to a constant and ϕ_1 and ϕ_2 to

$$\phi_1 = A \exp(\pm isx) \exp(-sz) \tag{27}$$

$$\phi_2 = A \exp(\pm isx) \exp(+sz) \tag{28}$$

These equations give the solution for the electrostatic problem, a solution that we will need for solving time dependent problems. Here the decay for up and down mapping is the same. This shows quite clearly that the difference in the attenuation in the previous cases is caused by the changing conductivity.

IV. APPLICATION OF THE SOLUTION TO AN IONOSPHERIC PROBLEM

For the application of eigenfunctions of a differential equation to a specific physical problem, we have to specify the boundary condition and the source of the potential

function. Here we impose the following boundary conditions:

1. The current flow is confined to an infinite long cylinder with a quadratic cross section as shown in Fig. 1.
2. Current with an average current density of $\bar{j} = 10^{-5} A/m^2$ is injected into the left half of the cylinder top and ejected out of the right half of the top surface.
3. No current should penetrate the side walls of the cylinder. For $x = 0$ and $x = 150$ km $\partial\phi/\partial x = 0$.
4. The bottom plate (earth surface) shall be an equipotential surface with the potential value 0.
5. The parallel conductivity λ_1 and the Pedersen conductivity λ_2 are given by

$$
\begin{aligned}
\lambda_1 &= \lambda_0 \exp(2kz) \\
\lambda_2 &= \lambda_1/(1+a \exp(2kz))^2
\end{aligned}
\qquad (4.1)
$$

The solution of the differential equation for steady state conditions has been given in the form of eigenfunctions in section II, equations (21) and (22). It remains to construct by superposition of these eigenfunctions with suitable adjusted constants A a solution that fits the boundary conditions.

First we accommodate the boundary condition 3. for the inpole function. $\partial\phi/\partial x = 0$ for $x = 0$ and $x = 150$ km. From (21)

$$\partial\phi_1/\partial x = A\, is \exp(\pm isx)\, f_1 = Asf_1\, i(\cos(sx) + i \sin(sx)) \quad (4.$$

The boundary condition 3, [illegible] fulfilled [illegible] the [illegible] part of (4.2) for

$$s = \frac{n\pi}{150\ \text{km}} \quad ; \quad n = 1, 2, \ldots \qquad (4.3)$$

Therefore

$$\frac{\partial \phi_1}{\partial x} = -A\ s\ f_1\ \sin(sx) \qquad (4.4)$$

and

$$\phi_1 = A\ f_1\ \cos(sx) \qquad (4.5)$$

The same boundary condition applies also to the outpole function, resulting in

$$\phi_2 = B\ f_2\ \cos(sx) \qquad (4.6)$$

with

$$\frac{\partial \phi_2}{\partial x} = -B\ s f_2\ \sin(sx) \qquad (4.7)$$

To enforce the boundary condition 4, $\phi = 0$ for $z = 0$, we apply the mirror image method and superimpose ϕ_1 and ϕ_2.

$$\phi = \phi_1 + \phi_2 = (A\ f_1 + B\ f_2)\ \cos(sx) \qquad (4.8)$$

for $z = 0$ follows from (19) and (20)

$$\left.\begin{aligned} f_{10} &= (1 + a)^{\frac{1}{2}\left(\sqrt{1 + (s/k)^2} + 1\right)} \\ f_{20} &= (1 + a)^{-\frac{1}{2}\left(\sqrt{1+(s/k)^2} - 1\right)} \end{aligned}\right\} \qquad (4.9)$$

$$\Lambda = -\frac{f_{20}}{f_{10}} B \qquad (4.10)$$

and

$$\phi = Bf_{20}(f_2/f_{20} - f_1/f_{10})\cos(sx) = B\ F(Z)\ \cos(sx) \qquad (4.11)$$

with

$$F(z) = f_2 - \frac{f_{20}}{f_{10}} f_1 \qquad (4.12)$$

The last boundary condition to be fulfilled is condition 2. To do this we have to calculate the average current density at the source for $z = H$.

We set

$$\left(\frac{dF}{dz}\right)_{z=H} = F'(H)$$

$$\bar{I}(H) = \bar{i}\lambda_2 Bs\ F'xin\ sx + \bar{k}\ \lambda_1\ B\ F'(H)\ \cos\ sx \qquad (4.13)$$

Since at the altitude of $z = H = 150$ km the Pedersen conductivity λ_2 is very much smaller than the parallel conductivity λ_1 we may neglect the horizontal against the vertical current flow component and write

$$I\ (H) = \lambda_1\ B\ F'(H)\ \cos(sx) \qquad (4.14)$$

This approximation is an expression of the magnetic field aligned current. The magnetic field is vertical and therefore only the vertical current component is significant.

We integrate the [illegible] (3) [illegible]

x [illegible] L [illegible], where [illegible]

divide by [illegible]

$$\hat{I}(H) = \frac{\lambda_1 B F'(H)}{sL} = \frac{\lambda_1 B F'(H)}{2}$$

The constant B is hereby determined.

$$B = \frac{s L \hat{I}}{\lambda_1 F'(H)} = \frac{2 \hat{I}}{\lambda_1 F'(H)} \tag{4.15}$$

V. NUMERICAL EVALUATION

1. Conductivities λ_1 and λ_2

$$\lambda_1 = \lambda_0 \exp(2kz)$$

$$\lambda_0 = 2 \times 10^{-14} \ 1/\Omega m$$

$$2k = \frac{\ln 10}{10 \ km}$$

At the ground the conductivity has the value of 2×10^{-14} $1/\Omega m$ and increases by a factor 10 every 10 km. At the source in 150 km altitude the conductivity λ_1 has the value of $\lambda_{1(H)} = 20 \ 1/\Omega m$.

$$\lambda_2 = \lambda_0 \frac{\exp(2kz)}{(1+a \exp(2kz))^2} \tag{5.1}$$

The factor "a" in the denominator is now adjusted in such a way that in the tropo- and stratosphere the Pedersen conductivity λ_2 is practically the same as the parallel conductivity λ_1. Approaching the lower ionosphere λ_2

separates from λ_1 reaches a maximum and finally decays with increasing altitude. The maximum is reached when

$$a = \exp(-2kz_m) \tag{5.2}$$

If we would like the maximum of λ_2 to be at 100 km altitude then

$$a = 10^{-10}.$$

Fig. 2 shows the conductivities of λ_1 and λ_2 given on the horizontal axis with a logarithmic scale as a function of altitude z on the vertical axis. The maximum of λ_2 is at 75 km altitude with $a = \sqrt{10} \times 10^{-8}$.

2. Potential Function

$$\phi = B\, F(z) \cos(sx)$$

$$B = -\pi\, \tilde{I} \,/\, 2\, \lambda_1(H)\, F'(H) = 11.4 \text{ MV}$$

Fig. 3 shows a plot of the equipotential lines with the zero potential as a straight vertical line in the middle connected to the ground. The lines to the left increase in steps of about 1 MV and the lines to the right decrease in steps of about -1 MV. It is remarkable that the equipotential lines in the ionosphere from 150 km to about 75 km altitude are practically vertical. Then they slant slightly to the left in the left half and to the right in the right half of Fig. 3 until they bend sharply in a horizontal direction as they approach the ground. This leads to a crowding of the equipotential lines in the surface layer of 10 to 20 km thickness and offers the opportunity to measure

the potential distribution in the ionosphere by balloon or airplane sounding of the atmospheric electric potentials in the lowest layer of the atmosphere.

Function ϕ/ϕ_H versus altitude z is plotted in Fig.4. The horizontal axis shows ϕ/ϕ_H from 0 to 1 and the vertical axis the altitude z from 0 to 150 km. In 10 km altitude we have already reached 80% of the final value of V. The shape of the curve is the same for any distance from the midpoint x = 75 km, only the final value changes with cos(sx) from the maximum value of about +10 MV.

3. Electric Field Components E_x and E_z

$$E_x = s\, B\, F(z)\, \sin sx$$

$$E_z = -B\, F'(z)\, \cos sx$$

The magnitude of the electric field components E_x and E_z versus altitude are shown in Fig. 5. The horizontal axis displays the fields E_x, E_z and the vertical axis the altitude z. The vertical field E_z has its maximum at the ground of about +2.3 kV/m and decreases rapidly with increasing altitude. At 10 km altitude the value is E_z = +40 V/m.

The horizontal field E_x is zero at the ground reaches 200 V/m at 10 km and about 220 V/m at 20 km altitude. From here on it keeps its value all the way up to 150 km altitude. This is indeed a very remarkable result. Since the

atmospheric fair weather field has no horizontal field component, the horizontal field component would be the best parameter to distinguish aurora from fair weather fields. This would become especially important if average aurora currents are only 1/10 or 1/100 as strong as assumed in this report. Horizontal electric fields of 22 or 2.2 V/m can still be measured with sufficient accuracy.

Another distinguishing characteristic is the phase shift between the horizontal and vertical component E_x and E_z. For $x = 0$ and $x = 150$ km the vertical component E_z has its maximum or minimum value of ± 250 V/m at 10 km altitude. At $x = 75$ km it passes through zero and changes polarity. The horizontal component E_x is zero at $x = 0$ and $x = 150$ km and has its maximum value of 200 V/m at $x = 75$ km where E_z passes through zero. An airplane flying at 10 km altitude from $x = 0$ to $x = 150$ km would record a field pattern as shown in Fig. 6. The horizontal axis gives here the flight path x at 10 km altitude and the vertical axis the field components E_x and E_z. E_x is displayed by a dashed and E_z by a solid line. It is most unlikely that such a field pattern could be caused by an atmospheric electric fair weather field. The constant fair weather field would appear as a zero shift of the E_z component which could easily be eliminated from the record.

4. <u>Current density components I_x and I_z</u>

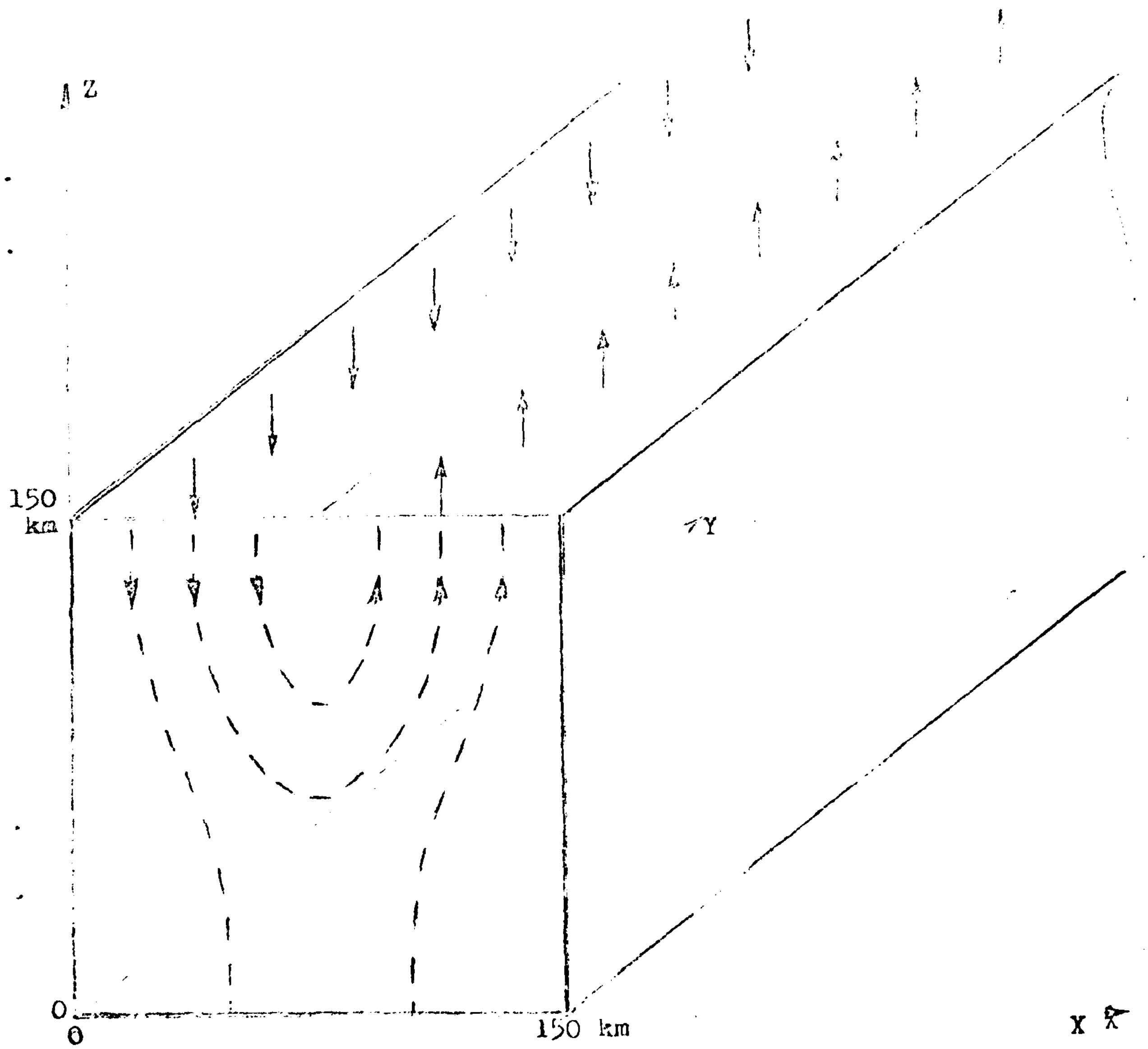
$$I_x = \lambda_2 E_x$$

$$I_z = \lambda_1 E_z$$

The current components I_x and I_z are even more remarkable than the field components E_x and E_z. Fig. 7 shows the current profiles with the vertical axis giving the altitude z and the horizontal axis I_x and I_z. Both components are weak in the tropo- and lower stratosphere. I_x increases rapidly from 50 km to 75 km altitude where it reaches its maximum value of almost 4×10^{-5} A/m^2. This is by a factor 4 higher than the average current density and by a factor 2.9 than the maximum current density at the source. From 75 to 100 km altitude the horizontal current density decreases again and becomes negligable in altitudes of 100 to 150 km. The vertical component I_z starts to increase from about 55 km reaches approximately its maximum value of $1.37.10^{-5}$ A/m^2 at 100 km and stays at this value up to 150 km altitude (field aligned current).

Fig. 8 gives the current profiles of I_x and I_z in the lower layer of the atmosphere from 0 to 30 km altitude. I_z is 4.7×10^{-11} A/m^2 at the ground. This is already higher by about a factor 20 than the fair weather current. It has doubled its value at about 20 km altitude and increases more rapidly with further increase of altitude. The horizontal component I_x starts with zero at the ground reaches the value of I_z at about 11 km altitude and increases much more rapidly than I_z for higher altitudes. Note, however, that that phase shift between E_x and E_z discussed above applies also to I_x and I_z.

It should not be difficult to design from the characteristic differences in the field and current pattern of the fair weather and aurora current flow an evaluation method for the data obtained from a network on the ground or from an airplane to separate the fair weather from the aurora fields or currents. It is obvious, however, from Fig. 7 and Fig. 8 that the current density I_z of the aurora currents is not constant with altitude. Since a constant current density with altitude is a necessary assumption to extrapolate the fair weather potential at flight altitude to the ionosphere to obtain the ionosphere potential of the atmospheric electric current the same extrapolation method would not be valid to determine potential differences in the ionosphere potential caused by solar or ionospheric events.



Current in- and out-flow of cylindrical box

Fig.1

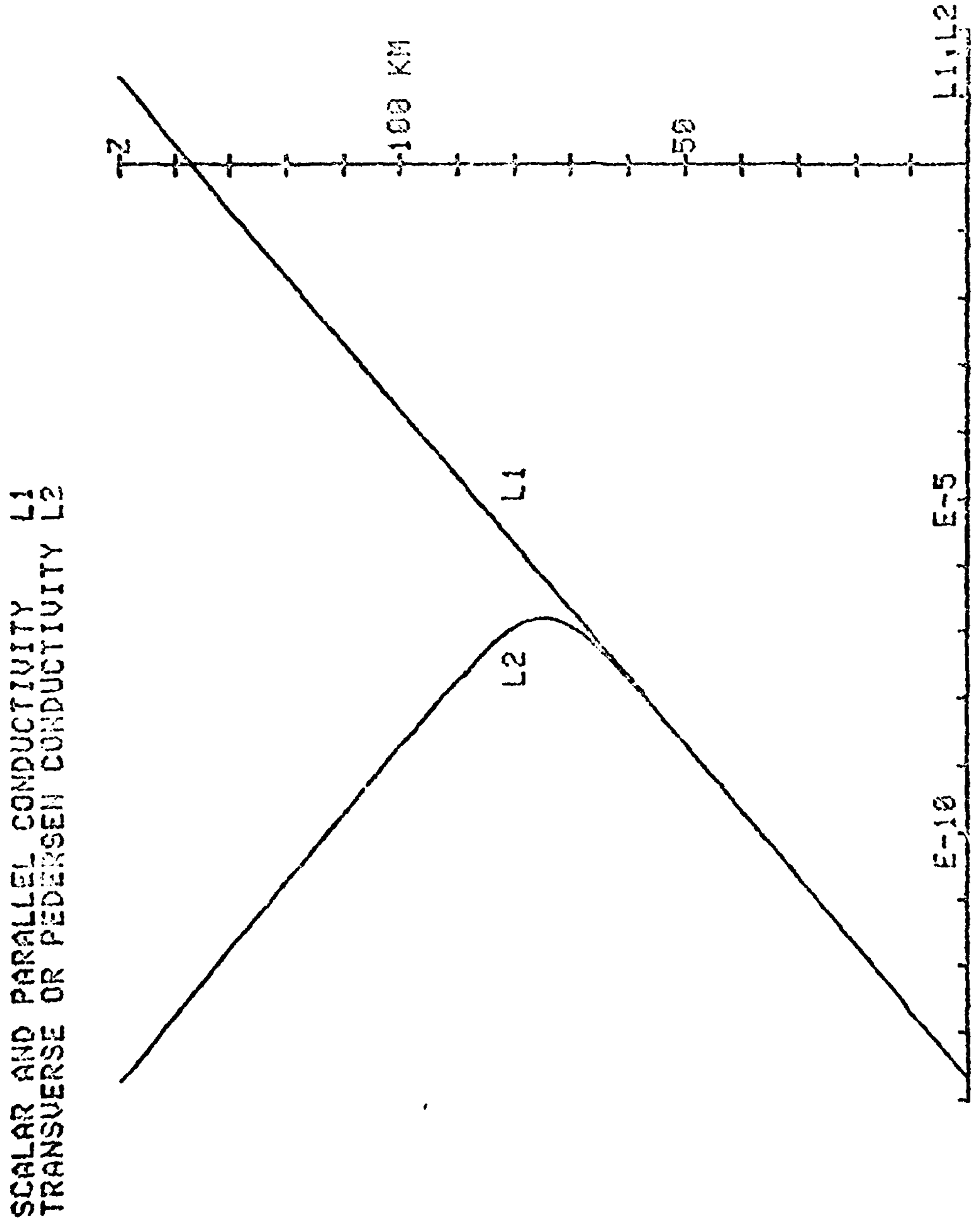
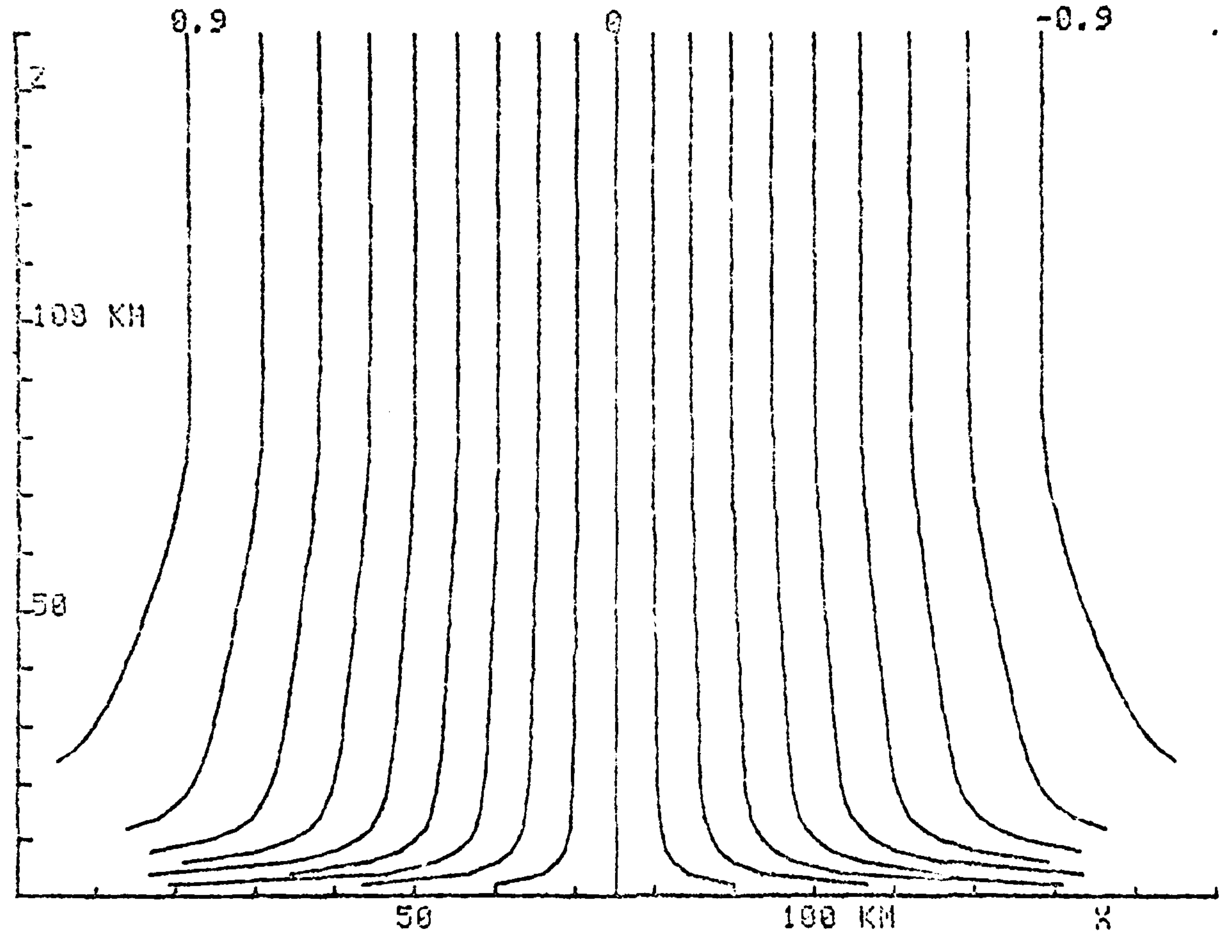

SCALAR AND PARALLEL CONDUCTIVITY L1
TRANSVERSE OR PEDERSEN CONDUCTIVITY L2
Z
100 KM
50
L1,L2
E-5
E-10
L1
L2

Fig.2

NORMALIZED EQUIPOTENTIAL LINES OF AURORA CURRENTS



NORMALIZED POTENTIAL FUNCTION Φ/Φ(H) VERSUS ALTITUDE Z

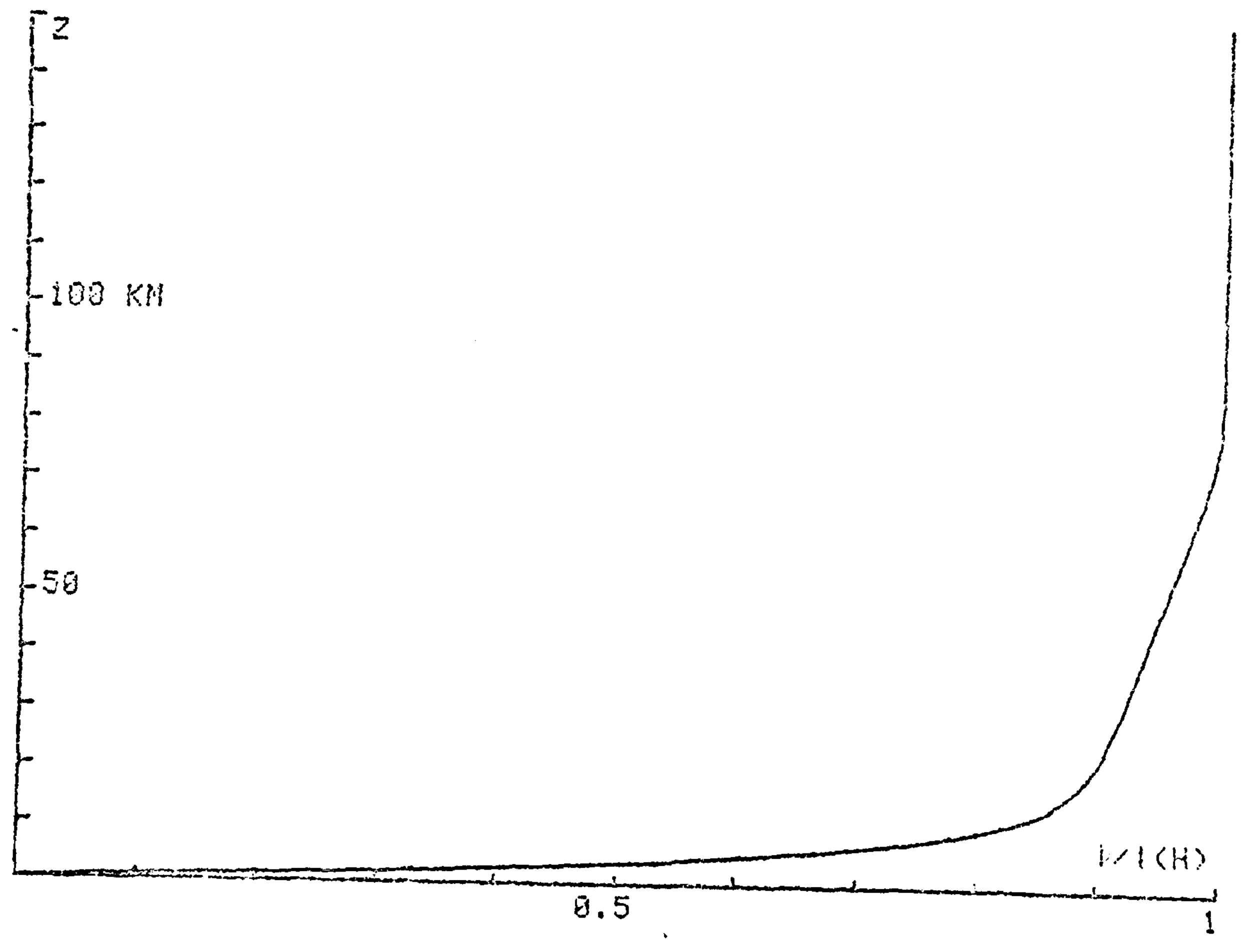


Fig.4

ELECTRIC FIELD COMPONENTS EX AND EZ IN KV/M
OF AURORA CURRENTS VERSUS ALTITUDE Z IN KM

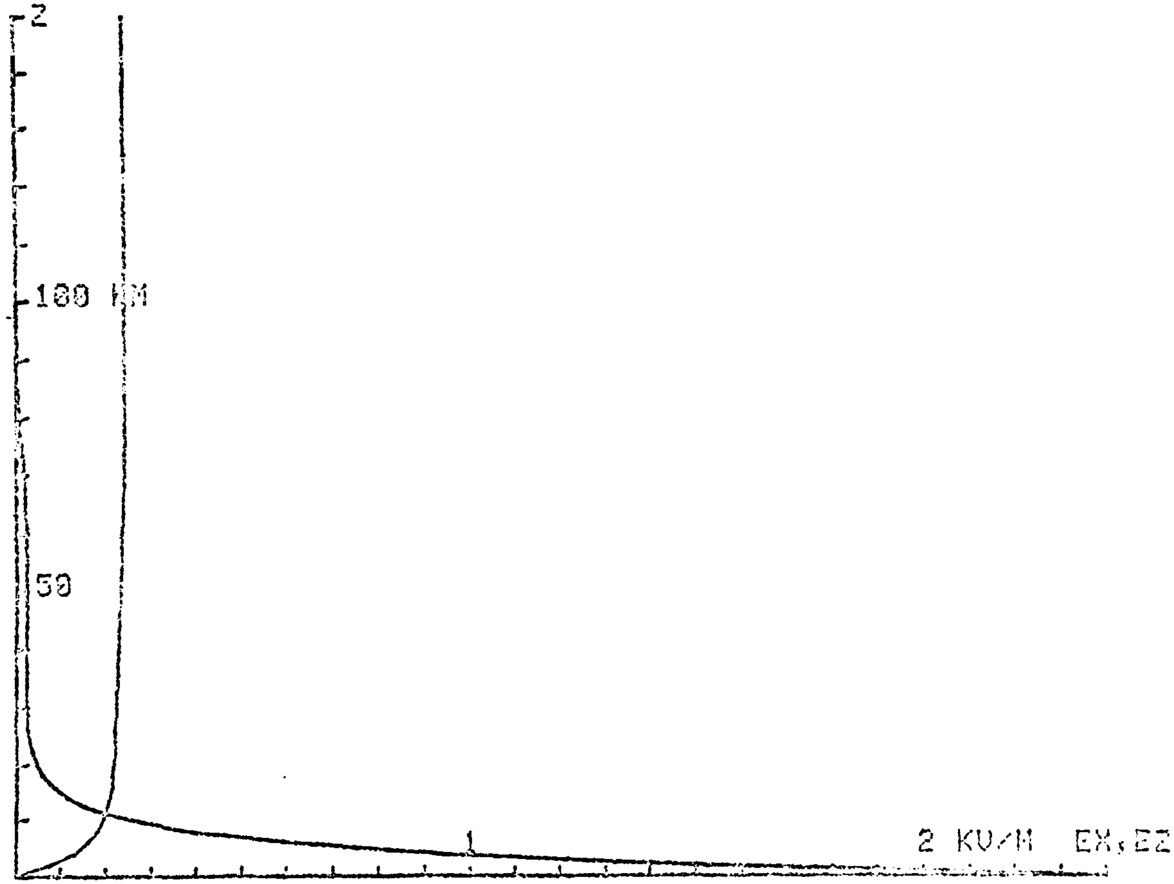


Fig. 5

EX AND EZ AT 10 KM ALTITUDE VERSUS HORIZONTAL DISTANCE X

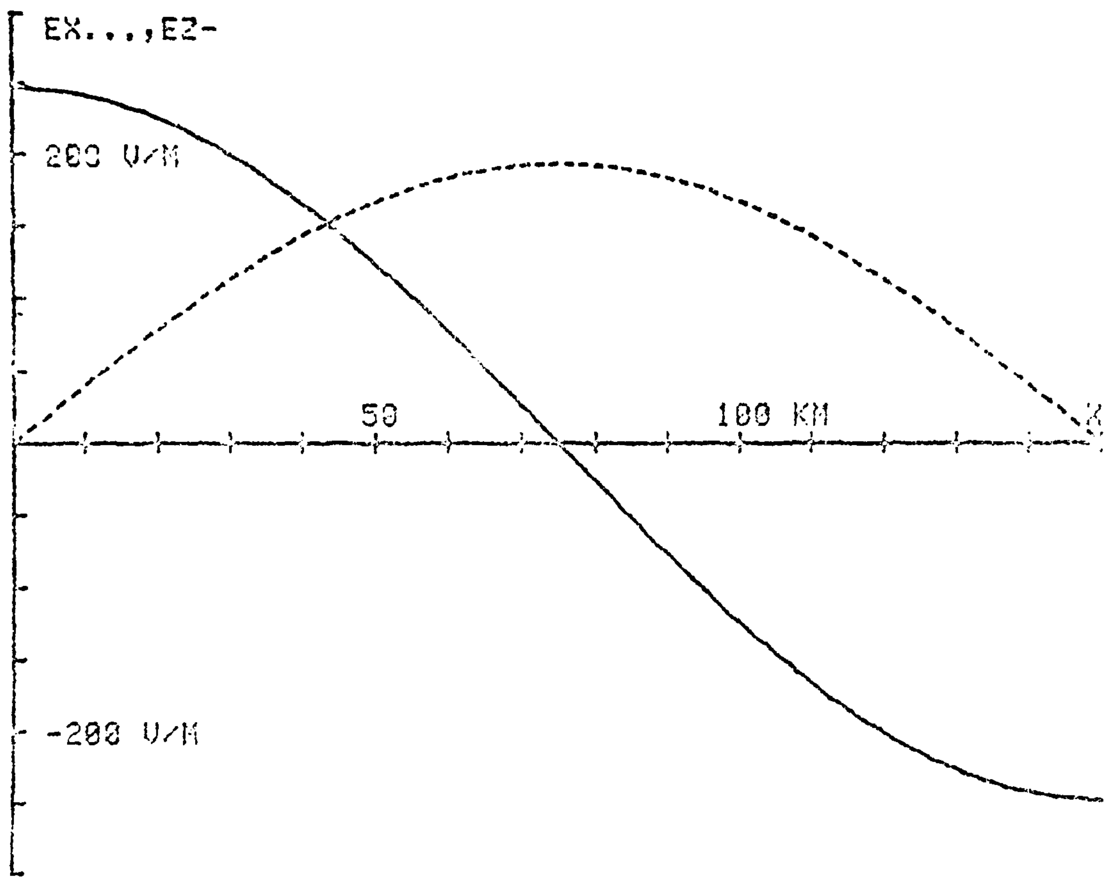
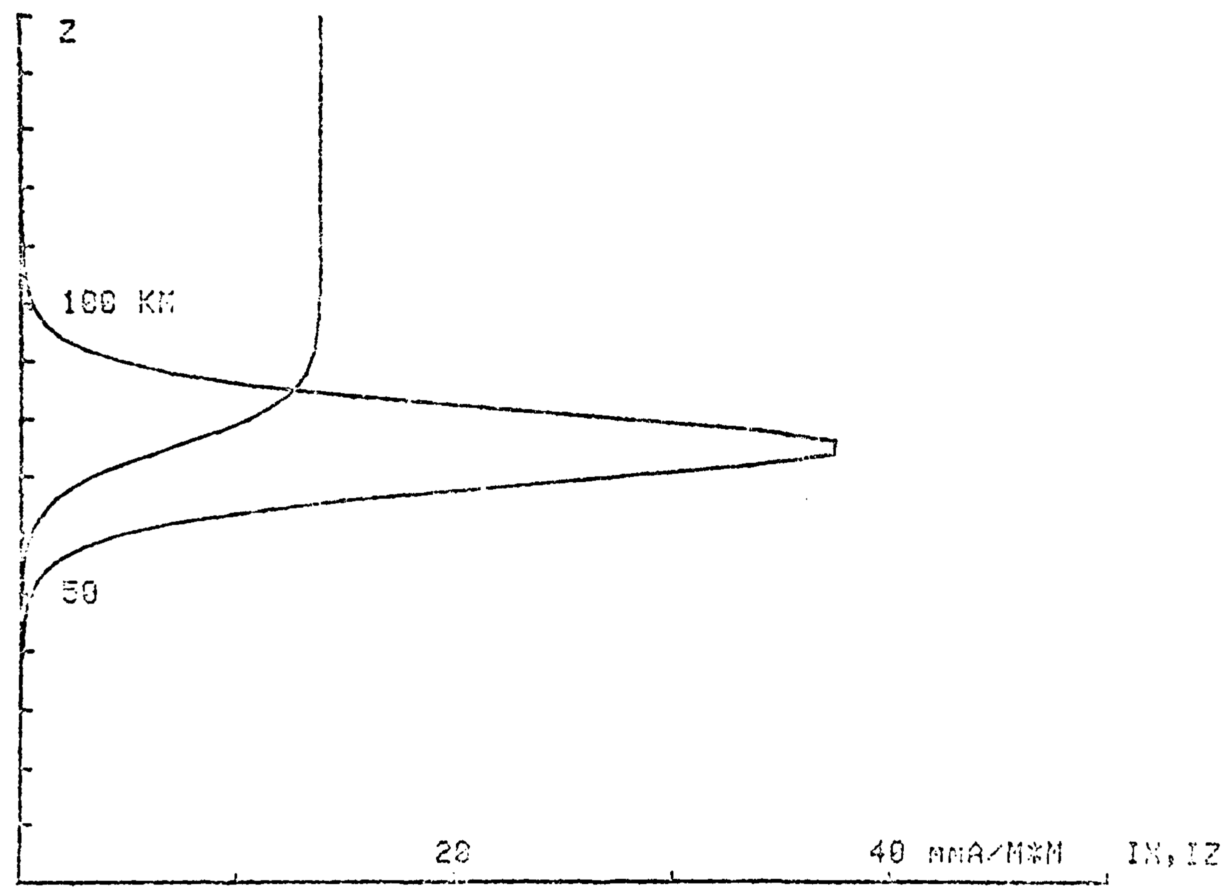


Fig. 6

CURRENT COMPONENTS IX,AND IZ VERSUS ALTITUDE Z
OF AURORA CURRENTS IN mmA/M*M



CURRENT DENSITIES IX AND IZ OF AURORA CURRENTS
BETWEEN 0 AND 30 KM ALTITUDE

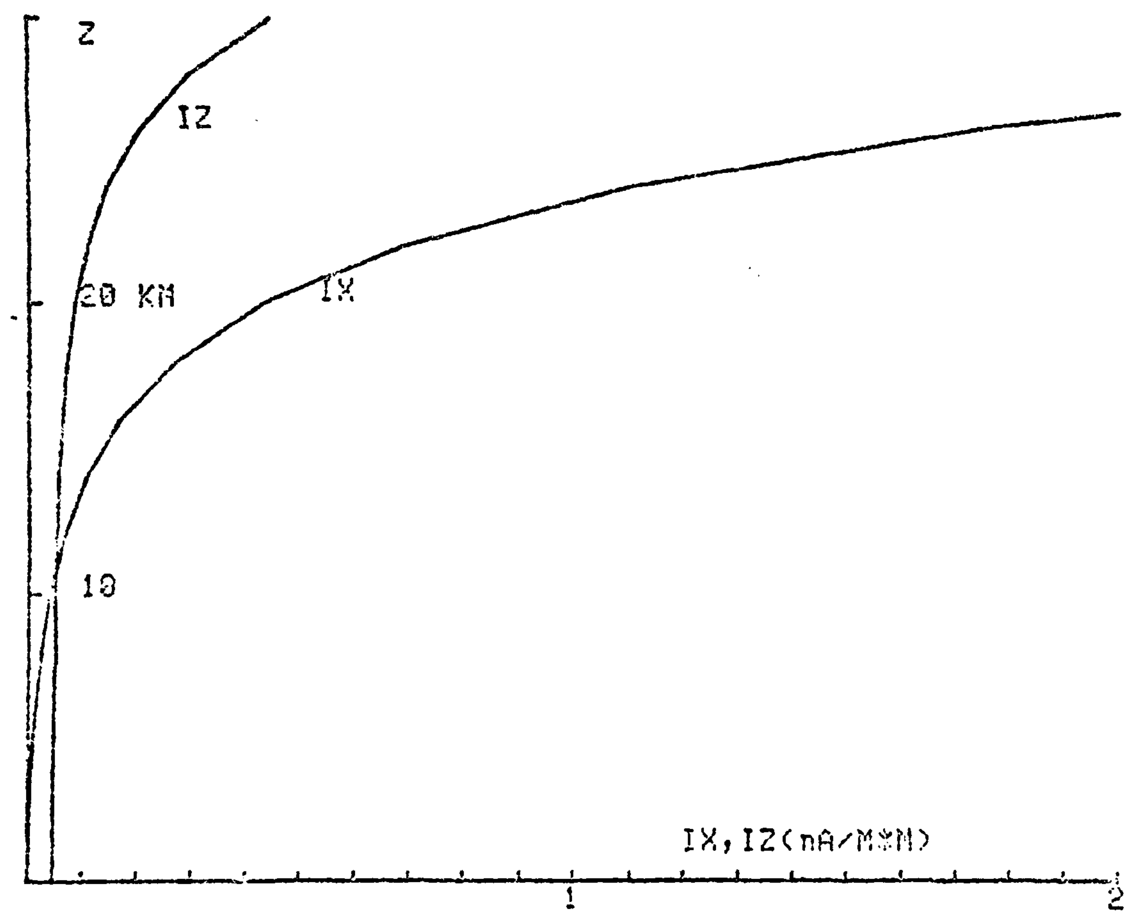


Fig. 8

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